New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2018 — Solutions

Due: 28th September 2018

1. *Suppose that a, b, c and d are four different integers. Explain why*

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

must be a multiple of 12.

Solution: Let $X = (a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$. There are 4 integers a, b, c, d but only 3 different possible remainders on division by 3. By the pigeonhole principle, two of the integers must have the same remainder on division by 3. This means that their difference must be a multiple of 3, and since their difference is a factor of X, this means that X must be a multiple of 3.

Now we only need to show that X is a multiple of 4. The four integers a, b, c, d might consist of two even and two odd numbers, or alternatively, some three or more of them might have the same parity (meaning they are all even, or all odd). Either way, amongst the numbers $\{a, b, c, d\}$ there are at least two pairs which have the same parity; that is, the must be at least two pairs of numbers that are both even or both odd. For each of these pairs their difference is even, and thus X is a multiple of 4.

Since X is a multiple of 3 and a multiple of 4, X must be a multiple of 12. □

2. *Find all pairs of integers (a, b) such that*

$$a^2 + ab - b = 2018.$$

Solution: The equation rearranges and factors as $(a-1)(a+b+1) = 2017$. Since 2017 is prime, it only has four factors: ± 1 and ± 2017. Since $(a-1)$ is a factor of 2017, this means that the only possibilities for a are 2018, 2, 0 or -2016. For each of these cases we can calculate b using $b = (2018 - a^2)/(a-1)$. So the only solutions are:

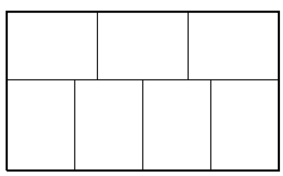
$$(a, b) = (2018, -2018), (2, 2014), (0, -2018), (-2016, 2014).$$

□

3. *Show that amongst any 8 points in the interior of a 7×12 rectangle, there exists a pair whose distance is less than 5.*

Note: The *interior* of a rectangle excludes points lying on the sides of the rectangle.

Solution: Partition the 7×12 rectangle into seven disjoint 3×4 rectangular tiles as shown.



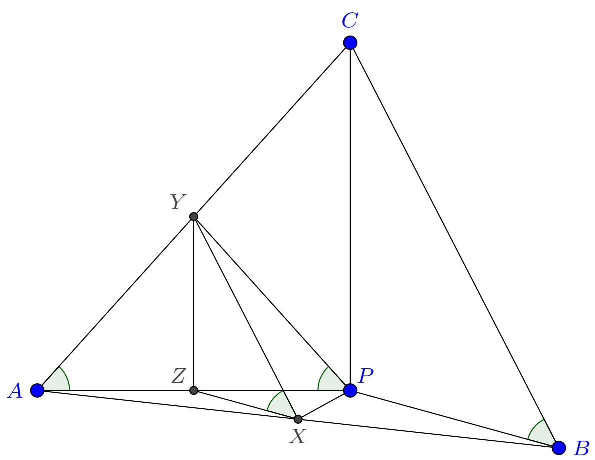
By the pigeonhole principle, there exists at least one 3×4 tile containing at least 2 points. The distance between two points in this tile is at most the length of the diagonal, which is 5. This distance can only equal 5 if the two points lie at opposite vertices of the tile. Since all the points are strictly in the interior of the 7×12 rectangle, and since any pair of opposite vertices of a tile includes at least one point on the perimeter of the 7×12 rectangle, this cannot happen. Therefore this distance is less than 5. □

4. *Let P be a point inside triangle ABC such that $\angle CPA = 90°$ and $\angle CBP = \angle CAP$. Prove that $\angle PXY = 90°$, where X and Y are the midpoints of AB and AC respectively.*

Solution: Let Z be the midpoint of AP. Since $\angle CPA = 90°$ and Y is the midpoint of AC, this means that Y is the circumcentre of $\triangle CAP$. So $YA = YP$ and hence $\triangle YAP$ is isosceles. Hence $\angle YAZ = \angle YPZ$. Notice that X, Y and Z are the midpoints of AB, AC and AP respectively. In other words, $\triangle XYZ$ is the dilation of $\triangle BCP$ by a factor of $1/2$ about point A. Hence triangles XYZ and BCP are similar. Therefore

$$\angle YXZ = \angle CBP = \angle CAP = \angle YAZ = \angle YPZ.$$

Since $\angle YXZ = \angle YPZ$, it follows that $XPYZ$ is a cyclic quadrilateral. Since $YZP = 90°$, this means that YP is a diameter of the circumcircle of $XPYZ$. Therefore $\angle PXY = 90°$ as required.



□

5. *Let a, b and c be positive real numbers satisfying*

$$\frac{1}{a+2019} + \frac{1}{b+2019} + \frac{1}{c+2019} = \frac{1}{2019}.$$

Prove that $abc \geq 4038^3$.

Solution: Let $x = \frac{a}{2019}$, let $y = \frac{b}{2019}$ and $z = \frac{c}{2019}$. Substituting this into the above equation yields:

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1.$$

If we multiply both sides by $(x+1)(y+1)(z+1)$, and expand and simplify, then we get $xyz = 2 + x + y + z$. From here, using the AM-GM inequality we can deduce that

$$\frac{xyz}{4} = \frac{2+x+y+z}{4} \geq \sqrt[4]{2xyz}.$$

This rearranges to give $xyz \geq 8$. Therefore $abc = 2019^3 xyz \geq 4038^3$ as required. □

6. *The intersection of a cube and a plane is a pentagon. Prove the length of at least one side of the pentagon differs from* 1 *metre by at least* 20 *centimetres.*

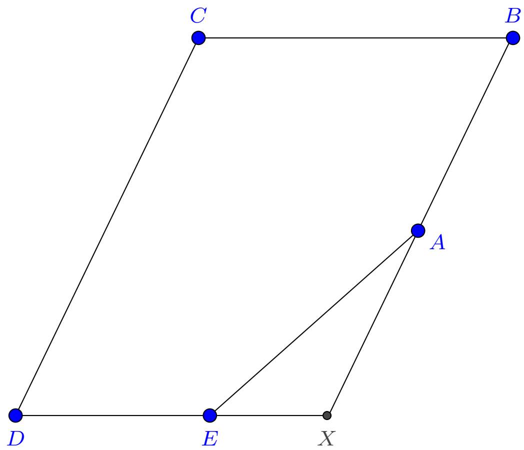
Solution: First note that any cross section of any convex shape is convex — so the pentagon must be convex. Also, the intersection of any plane with two opposite faces of the cube must be a pair of parallel lines (because opposite faces of the cube are parallel planes).

For the cross section to be a pentagon, the plane must intersect exactly 5 of the 6 sides of the cube. In particular, there is only one face which it does not intersect. This means that it intersects two different pairs of opposite sides of the cube. Therefore the pentagon has two different pairs of parallel sides. We conclude that

- the pentagon is convex, and
- the pentagon has two pairs of parallel sides.

This is all we need to know about the pentagon.

Now let the pentagon be $ABCDE$. Without loss of generality $AB||CD$ and $BC||DE$. Assume, for the sake of contradiction that all the sides of the pentagon are between 80 and 120 (all lengths are in cm). Now let X be the the intersection of AB and DE, so that $XBCD$ is a parallelogram.



We can now compute $XA = XB - AB = CD - AB < 120 - 80 = 40$, and so $XA < 40$. Similarly we must have $XE < 40$ also. By the triangle inequality we get

$$EA < EX + XA < 40 + 40 = 80.$$

However this contradicts the assumption that all side lengths of the pentagon are more than 80. □

7. *Let N be the number of ways to colour each cell in a 2×50 rectangle either red or blue such that each 2×2 block contains at least one blue cell. Show that N is a multiple of 3^{25}, but not a multiple of 3^{26}.*

Solution: We will let a_n denote the number of ways to colour each cell in a $2 \times n$ rectangle either red or blue such that no 2×2 block is entirely coloured red. Note that $a_1 = 2^2 = 4$ and $a_2 = 2^4 - 1 = 15$. Let's say that a 2×1 column is *hot* if both cells in the column are red, otherwise we say that the column is *cold*. Note that there are 3 different types of cold column.

We will count colourings according to the number of hot columns they contain. To do this we will use the following lemma:

Lemma. *Suppose that the cells of a $1 \times n$ rectangle are coloured black and white in such a way that no two black cells are adjacent. The number of such colourings with exactly k black cells is $\binom{n-k+1}{k}$.*

Proof. Given an allowable colouring with k black cells, there must be a white cell immediately to the right of each of the leftmost $k-1$ black cells. Deleting these gives a colouring of a $1 \times (n-k+1)$ rectangle, in which there are k black cells but no other restriction on how the cells may be coloured. Conversely, given an arbitrary colouring of a $1 \times (n-k+1)$ rectangle that has exactly k black cells, we may obtain an allowable colouring of a $1 \times n$ rectangle that has exactly k black cells by inserting a white cell immediately to the right of each of the leftmost $k-1$ black cells.

This establishes a bijection between allowable colourings of a $1 \times n$ rectangle with exactly k black cells, and arbitrary colourings of a $1 \times (n-k+1)$ rectangle with exactly k black cells. Since there are clearly $\binom{n-k+1}{k}$ of the latter, the lemma is proved. □

Returning now to the problem, we claim that there are $3^{n-k}\binom{n-k+1}{k}$ allowable colourings of a $2 \times n$ rectangle with exactly k hot columns. To see this, note that the hot columns may not be next to each other, so by the lemma there are $\binom{n-k+1}{k}$ ways to choose which columns are hot and which are cold; and then the cold columns come in three types, so there are 3^{n-k} ways to colour the cold columns. There can be at most $\lfloor \frac{n+1}{2} \rfloor$ hot columns, so

$$a_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} 3^{n-k}\binom{n-k+1}{k}.$$

In particular,

$$N = a_{50} = \sum_{k=0}^{25} 3^{50-k}\binom{51-k}{k} = 3^{25}\sum_{k=0}^{25} 3^{25-k}\binom{51-k}{k}.$$

This shows that N is divisible by 3^{25}. To see that it is not divisible by 3^{26}, separate out the $k = 25$ term to write N in the form

$$N = 3^{26}\sum_{k=0}^{24} 3^{24-k}\binom{51-k}{k} + 3^{25}\binom{26}{25}.$$

Since $\binom{26}{25} = 26$ is not divisible by 3 this completes the proof.

Alternate solution. Consider the left-most column in a valid colouring of the $2 \times n$ rectangle.

- If the leftmost column is cold, then there are a_{n-1} different ways to colour the rest of the rectangle. Since there are 3 different types of cold column, we can count $3a_{n-1}$ different valid colourings in this case.
- If the leftmost column is hot, then the second-left-most column must be cold, and there are a_{n-2} different ways to colour the rest of the rectangle. Since there are 3 different types of cold column, we can count $3a_{n-2}$ different valid colourings in this case.

It follows that

$$a_n = 3a_{n-1} + 3a_{n-2}.$$

We will now show for each positive integer k, that both a_{2k} and a_{2k+1} are multiples of 3^k. The base case ($k = 1$) is trivial because both $a_2 = 15$ and $a_3 = 57$ are multiples of 3. For the inductive step, let $a_{2k} = 3^k x$ and $a_{2k+1} = 3^k y$. The recurrence gives us $a_{2(k+1)} = 3(3^k x + 3^k y) = 3^{k+1}(x+y)$, and so $a_{2(k+1)}$ is a multiple of 3^{k+1}. Furthermore:

$$a_{2(k+1)+1} = 3(3^{k+1}(x+y) + 3^k y) = 3^{k+1}(3x+4y).$$

This completes our induction. The case $k = 25$ tells us that 3^{25} is a factor of $N = a_{50}$.

To show that N is not divisible by 3^{26} we now consider the sequence b_n defined by

$$b_{2k} = a_{2k}/3^k, \qquad b_{2k+1} = a_{2k+1}/3^k.$$

Our calculations above give

$$b_{2(k+1)} = b_{2k} + b_{2k+1}, \qquad b_{2(k+1)+1} = 3b_{2k} + 4b_{2k+1}.$$

Considering $b_{2(k+1)+1}$ modulo 3 we see that

$$b_{2(k+1)+1} \equiv 4b_{2k+1} \equiv b_{2k+1} \pmod 3,$$

and since $b_1 = a_1 = 4 \equiv 1 \pmod 3$ it follows that $b_{2k+1} \equiv 1 \pmod 3$ for all k. This means that 3^k is the highest power of 3 dividing a_{2k+1} for all k.

Looking now at $b_{2(k+1)}$ modulo 3 we have

$$b_{2(k+1)} = b_{2k} + b_{2k+1} \equiv b_{2k} + 1 \pmod 3.$$

Since $b_2 = a_2/3 = 15/3 = 5 \equiv 2 \pmod 3$, it follows that $b_{2k} \equiv 0 \pmod 3$ if and only if $k \equiv 2 \pmod 3$. Equivalently, 3^k is the highest power of 3 dividing a_{2k} except when $k \equiv 2 \pmod 3$. The case $k = 25$ then tells us that 3^{25} is the highest power of 3 dividing $N = a_{50}$, so N is not divisible by 3^{26}. □

8. *Let λ be a line and let M, N be two points on λ. Circles α and β centred at A and B respectively are both tangent to λ at M, with A and B being on opposite sides of λ. Circles γ and δ centred at C and D respectively are both tangent to λ at N, with C and D being on opposite sides of λ. Moreover A and C are on the same side of λ. Prove that if there exists a circle tangent to all circles $\alpha, \beta, \gamma, \delta$ containing all of them in its interior, then the lines AC, BD and λ are either concurrent or parallel.*

Solution: Assume that there exists a circle which is internally tangent to $\alpha, \beta, \gamma, \delta$. Let this circle be ρ, and let R be the centre of ρ. Without loss of generality R is on the same side of λ as A and C. Let the perpendicular distance from R to λ be h. Furthermore, let a, b, c, d, r be the radii of circles $\alpha, \beta, \gamma, \delta, \rho$ respectively. Since ρ is tangent to both α and β, this means that $AR = r - a$ and $BR = r - b$. Now construct point X on line AB such that RX is perpendicular to AB (see Figure 1).

Then RX is parallel to λ, and thus $AX = a - h$. Pythagoras' Theorem in triangle AXR gives us

$$RX^2 = RA^2 - AX^2 = (r-a)^2 - (a-h)^2 = r^2 - h^2 - 2a(r-h).$$

Similarly, we can compute $RB = r - b$ and $BX = b + h$, and so Pythagoras' Theorem in triangle BXR gives us

$$RX^2 = RA^2 - AX^2 = (r-b)^2 - (b+h)^2 = r^2 - h^2 - 2b(r+h).$$

Equating the right-hand sides of the above equations gives us $a(r-h) = b(r+h)$ and therefore $a/b = (r+h)/(r-h)$. Using a similar argument we get $c/d = (r+h)/(r-h)$, and thus we can deduce

$$\frac{a}{b} = \frac{c}{d}.$$

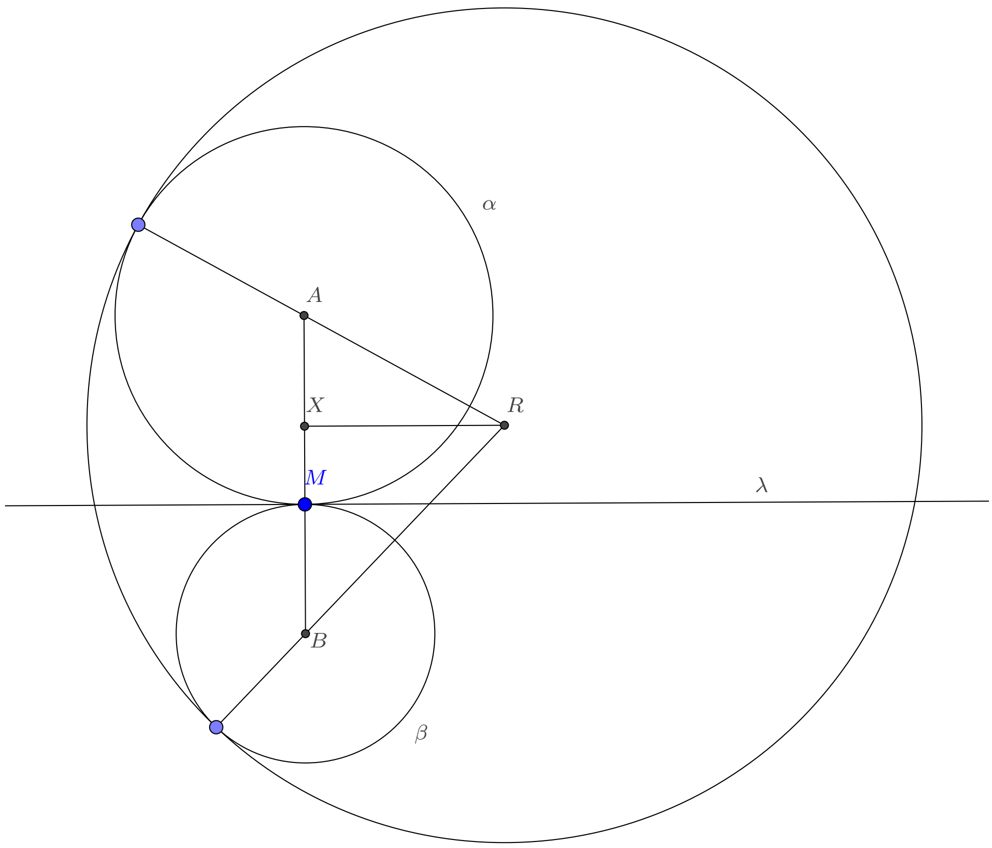


Figure 1: Diagram for Problem 8.

Now if $a = c$ then we would have $b = d$, and so both AC and BD are parallel to λ. Otherwise let E be the point of intersection of AC with λ. Since triangles AME and BNE are similar, we get

$$\frac{ME}{NE} = \frac{AM}{BN} = \frac{a}{b} = \frac{c}{d} = \frac{CM}{DN}.$$

This implies that triangles CME and DNE are similar, and so C, D and E are collinear. □

9. *Let x, y, p, n, k be positive integers such that*

$$x^n + y^n = p^k.$$

Prove that if $n > 1$ is odd, and p is an odd prime, then n is a power of p.

Solution: We give a proof by contradiction. Let us assume that (x, y, p, n, k) is a tuple of positive integers satisfying $x^n + y^n = p^k$ such that $n > 1$ is odd, p is an odd prime, and n is not a power of p. Furthermore, let us assume that (x, y, p, n, k) is the *smallest* such tuple — assume that there is no other such tuple with a smaller value of $x+y+p+n+k$.

Now if n were composite, then we could write $n = mr$ with $1 < m < n$ such that m is not a multiple of p. In this case we would have

$$x^n + y^n = (x^m + y^m)(x^{(r-1)m} - x^{(r-2)m}y^m + \cdots + y^{(r-1)m}) = p^k.$$

Therefore $x^m + y^m = p^{k'}$ would be a power of p with $k' < k$. This would mean that (x, y, p, m, k') would be a smaller tuple than (x, y, p, n, k), which is a contradiction. Therefore n cannot be composite, which means that n must be a prime different to p. In particular, this means that n is not a multiple of p.

Now if either one of x or y were a multiple of p, then the other one would have to be too, and so

$$x^n + y^n = p^n\left((x/p)^n + (y/p)^n\right).$$

If this were the case then $(x/p, y/p, p, n, k)$ would be a smaller tuple than (x, y, p, n, k). Hence we can assume that neither x nor y is a multiple of p.

Now consider the factorisation:

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots + y^{n-1}).$$

This means that $x + y$ is a power of p, and since both are positive integers, we know $y \equiv -x \pmod p$. Considering the other factor modulo p gives us

$$\begin{aligned} &x^{n-1} - x^{n-2}y + x^{n-3}y^2 - x^{n-4}y^3 + \cdots + y^{n-1} \\ &\qquad \equiv x^{n-1} - x^{n-2}(-x) + x^{n-3}(-x)^2 - x^{n-4}(-x)^3 + \cdots + (-x)^{n-1} \\ &\qquad = nx^{n-1}. \end{aligned}$$

Since neither x nor n are multiples of p, this factor is not a multiple of p, which is a contradiction. This completes the proof □

10. *Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that*

$$f(x)f(y) = f(xy + 1) + f(x - y) - 2$$

for all $x, y \in \mathbb{R}$.

Solution: Substituting $y = 0$ into the equation gives us $(f(0) - 1)f(x) = f(1) - 2$. So if $f(0) \neq 1$, then $f(x) = (f(1) - 2)/(f(0) - 1)$ and so the function would be constant. However, since $\lambda^2 = 2\lambda - 2$ has no real solutions, the function cannot be constant. Therefore $f(0) = 1$, which in turn implies that $f(1) = 2$. Also note that f must be an even function because

$$f(x - y) = f(x)f(y) - f(xy + 1) + 2 = f(y - x).$$

Now set $f(x) = 1 + x^2 + g(x)$, for some even function $g : \mathbb{R} \longrightarrow \mathbb{R}$, with $g(0) = g(1) = 0$. The functional equation given in the problem now simplifies to

$$g(x)g(y) + (x^2 + 1)g(y) + (y^2 + 1)g(x) = g(xy + 1) + g(x - y) \qquad \forall\, x, y \in \mathbb{R}. \qquad (1)$$

For each real number x, let us define $\mathcal{P}(x)$ to be the following (two-sided) sequence:

$$\mathcal{P}(x) = \{\ldots, g(x - 2), g(x - 1), g(x), g(x + 1), g(x + 2), \ldots\}.$$

If we substitute $y = 1$ into equation (1), we get $g(x + 1) - g(x) = g(x) - g(x - 1)$. This means that $\mathcal{P}(x)$ is always an arithmetic progression. We can also substitute $y = -x$ and $y = x$ to get

$$\begin{aligned} g(x^2 + 1) + g(0) &= g(x)^2 + 2(x^2 + 1)g(x) \\ \text{and} \qquad g(1 - x^2) + g(2x) &= g(x)g(-x) + (x^2 + 1)(g(x) + g(-x)). \end{aligned}$$

Since g is even, the right-hand sides of the above two equations are equal. Equating the left-hand sides gives us

$$g(x^2+1) - g(x^2-1) = g(2x) \qquad \forall x \in \mathbb{R}. \tag{2}$$

Now let us assume for the sake of contradiction that there exists some real a such that $g(a) \neq 0$. By Equation (2) this means that the arithmetic progression $\mathcal{P}(a^2/4)$ has a non-zero constant difference. Hence the value of $g(a^2/4+i)$ is arbitrarily large (in both the positive and negative directions) as i ranges over all integers. Hence there exists some real number $b = a^2/4+i$ such that $g(b) < 8$. By Equation (2) this means that $\mathcal{P}(b^2/4)$ is an arithmetic progression with common difference less than -50. Therefore there exists some $c = b^2/4+i$ such that

$$1 < c < 2 \qquad \text{and} \qquad g(c-1) - 4 > g(c) > g(c+1) + 4.$$

Now substituting $x = \sqrt{c}$ into Equation (2) we get $g(2\sqrt{c}) = g(c+1) - g(c-1) < -8$. Therefore $2\sqrt{c}$ is a real number between 2 and $2\sqrt{2}$, such that $g(2\sqrt{c}) < -8$. Now let d be the real number such that $d^2+1 = 2\sqrt{c}$, and substitute $x = y = d$ into Equation (1) to get

$$\big(g(d) + d^2 + 1\big)^2 = g(d^2+1) + (d^2+1)^2 = g(2\sqrt{c}) + \big(2\sqrt{c}\big)^2 < -8 + 8 = 0.$$

This is a contradiction. We conclude that g is identically 0, and hence that $f(x) = x^2+1$ for all x. □

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