New Zealand Mathematical Olympiad Committee

NZMO Round One 2022 — Instructions

Submissions due date: 1st August

The **New Zealand Mathematical Olympiad** (NZMO) consists of two rounds:

Round One (the NZMO1): A take home exam (the following set of 8 problems). Solutions are to be submitted by 1st *August* (21:00 NZTZ).

Round Two (the NZMO2): A three hour supervised exam in September.

Awards for the NZMO (gold/silver/bronze/honourable mention) will be made based on the results of the NZMO2. Scores will not be announced for either round. Participants in the NZMO1 will only receive an indication of whether they have been invited to participate in the NZMO2; participants in the NZMO2 will just be told the level of their award, if any.

In addition, the results of both rounds of the NZMO will be used to select about 25 students to participate in the NZMOC training camp, to be held in January 2023. Only students who are New Zealand citizens or permanent residents, and who will still be enrolled in intermediate or high school in 2023, are eligible for NZMOC training camp selection. Only students who attend this camp are eligible for selection to represent . The training camp is the first step in the selection process to choose the team of six students to represent New Zealand at the 2023 International Mathematical Olympiad, (IMO).

General instructions:

- Participation in the NZMO is open to any student enrolled in the New Zealand education system, at secondary school level or below.
- The NZMO is an individual competition. Participants must work on the problems entirely on their own, without assistance from anyone else or any kind of software.
- Electronic devices may not be used to assist in solving the problems. This includes but is not limited to calculators, computers, tablets, smart phones and smart watches.
- The internet may not be used, except as a reference for looking up definitions.
- Although some problems may seem to require only a numerical answer, in order to receive full credit for the problem a complete justification must be provided.
- Submitting rough working and/or partial solutions cannot negatively affect the score awarded. We encourage you to submit all rough work and any partial solutions.
- It is forbidden to discuss the problems especially online until the official solutions are posted on the NZMOC website. Typically this will be 3 days after the submission deadline.
- Please address **all** queries to Dr Ross Atkins, info@mathsolympiad.org.nz

The NZMO1 submission url is: https://www.mathsolympiad.org.nz/nzmo1_submission. If you are having trouble loading the submission form, please try logging out of your google account, or opening the form in an incognito window. All your solutions and partial solutions should be submitted as a **single document** in PDF format. Please complete the submission form **carefully**, especially your contact details. If you have any difficulties with the submission form, contact Dr Ross Atkins (info@mathsolympiad.org.nz).

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NZMO Round One 2022 — Problems

Submissions due date: 1^{st} August

- There are 8 problems.
- Read and follow the "General instructions" accompanying these problems.
- If any clarification is required, please contact Dr Ross Atkins (info@mathsolympiad.org.nz).

Problems

1. $ABCD$ is a rectangle with side lengths $AB = CD = 1$ and $BC = DA = 2$. Let M be the midpoint of AD. Point P lies on the opposite side of line MB to A, such that triangle MBP is equilateral. Find the value of $\angle PCB$.
2. Is it possible to pair up the numbers $0, 1, 2, 3, \ldots, 61$ in such a way that when we sum each pair, the product of the 31 numbers we get is a perfect fifth power?
3. Find all real numbers x and y such that
$$x^2 + y^2 = 2,$$
$$\frac{x^2}{2-y} + \frac{y^2}{2-x} = 2.$$
4. On a table, there is an empty bag and a chessboard containing exactly one token on each square. Next to the table is a large pile that contains an unlimited supply of tokens. Using only the following types of moves what is the maximum possible number of tokens that can be in the bag?
   - **Type 1:** Choose a non-empty square on the chessboard that is not in the rightmost column. Take a token from this square and place it, along with one token from the pile, on the square immediately to its right.
   - **Type 2:** Choose a non-empty square on the chessboard that is not in the bottom-most row. Take a token from this square and place it, along with one token from the pile, on the square immediately below it.
   - **Type 3:** Choose two adjacent non-empty squares. Remove a token from each and put them both into the bag.
5. A round-robin tournament is one where each team plays every other team exactly once. Five teams take part in such a tournament getting: 3 points for a win, 1 point for a draw and 0 points for a loss. At the end of the tournament the teams are ranked from first to last according to the number of points.
   (a) Is it possible that at the end of the tournament, each team has a different number of points, and each team except for the team ranked last has exactly two more points than the next-ranked team?
   (b) Is this possible if there are six teams in the tournament instead?

6. Let a positive integer n be given. Determine, in terms of n, the least positive integer k such that among any k positive integers, it is always possible to select a positive even number of them having sum divisible by n.

7. Let M be the midpoint of side BC of acute triangle ABC. The circle centered at M passing through A intersects the lines AB and AC again at P and Q, respectively. The tangents to this circle at P and Q meet at D. Prove that the perpendicular bisector of BC bisects segment AD.

8. Find all real numbers x such that $-1 < x \leq 2$ and

$$\sqrt{2-x} + \sqrt{2+2x} = \sqrt{\frac{x^4+1}{x^2+1} + \frac{x+3}{x+1}}.$$