2019 NZMO Round 2

Solutions

1. *A positive integer is called* sparkly *if it has exactly 9 digits, and for any n between 1 and 9 (inclusive), the n^{th} digit is a positive multiple of n. How many positive integers are sparkly?*

   Solution: For each $n = 1, 2, \ldots, 9$ there are $\lfloor 9/n \rfloor$ different possibilities for the n^{th} digit. For example there are $\lfloor 9/2 \rfloor = 4$ possible choices for the second digit (these being 2, 4, 6 and 8). Therefore the answer is

   $$\left\lfloor \frac{9}{1} \right\rfloor \times \left\lfloor \frac{9}{2} \right\rfloor \times \left\lfloor \frac{9}{3} \right\rfloor \times \left\lfloor \frac{9}{4} \right\rfloor \times \left\lfloor \frac{9}{5} \right\rfloor \times \left\lfloor \frac{9}{6} \right\rfloor \times \left\lfloor \frac{9}{7} \right\rfloor \times \left\lfloor \frac{9}{8} \right\rfloor \times \left\lfloor \frac{9}{9} \right\rfloor$$

   which equals $9 \times 4 \times 3 \times 2 \times 1 \times 1 \times 1 \times 1 \times 1 = 216$. □

2. *Let X be the intersection of the diagonals AC and BD of convex quadrilateral $ABCD$. Let P be the intersection of lines AB and CD, and let Q be the intersection of lines PX and AD. Suppose that $\angle ABX = \angle XCD = 90°$. Prove that QP is the angle bisector of $\angle BQC$.*

   Solution: First note that quadrilateral $ABCD$ is cyclic because $\angle ABD = \angle ACD = 90°$. Also, since $AP \perp DX$ and $DP \perp AX$, we see that X is the orthocentre of triangle APD. Hence $PX \perp AD$. Therefore quadrilaterals $ABXQ$ and $QXCD$ are cyclic (opposite angles are supplementary). Now we perform a simple angle chase

   $$\angle XQB = \angle XAB = \angle CAB = \angle CDB = \angle CDX = \angle CQX.$$

   Since $\angle XQB = \angle CQX$, it follows that QX is the angle bisector of $\angle CQB$ as required. □

3. *Let a, b and c be positive real numbers such that $a + b + c = 3$. Prove that*

   $$a^a + b^b + c^c \geq 3.$$

   Solution: We start with a general fact about any positive real number x. There are two cases: either $x \geq 1$ or $x < 1$.

   - If $x \geq 1$ then $x^p \geq x^q$ for any $p > q$. Substituting $p = x$ and $q = 1$ gives us $x^x \geq x^1 = x$.
   - If $x < 1$ then $x^p < x^q$ for any $p > q$. Substituting $p = 1$ and $q = x$ gives us $x = x^1 < x^x$.

   In either case we get $x^x \geq x$ for all positive real numbers x. Applying this fact for a, b and c gives us

   $$a^a + b^b + c^c \geq a + b + c = 3$$

   as required. □

4. *Show that for all positive integers k, there exists a positive integer n such that $n2^k - 7$ is a perfect square.*

   Solution: Proof by induction on k. For the base cases ($k \le 3$) we can simply choose $n = 2^{3-k}$ to get $n2^k - 7 = 2^3 - 7 = 1^2$. For the inductive step let $k \ge 3$ and assume there exist integers a and n such that
$$a^2 = n2^k - 7.$$
   We will now endeavour to find integers b and m such that $b^2 = m2^{k+1} - 7$. To do this we have two cases:

   - If n is even then choose $b = a$ and $m = n/2$. Thus $b^2 = (2m)2^k - 7 = m2^{k+1} - 7$, as required.
   - If n is odd, then note that a must also be odd. Let $n = 2x + 1$ and let $a = 2y + 1$. Now consider $(a + 2^{k-1})^2$.
$$\begin{aligned}(a + 2^{k-1})^2 &= a^2 + 2^k a + 2^{2k-2}\\ &= \left(n2^k - 7\right) + 2^k a + 2^{2k-2}\\ &= \left((2x+1)2^k - 7\right) + 2^k(2y+1) + 2^{2k-2}\\ &= \left(x + y + 1 + 2^{k-3}\right)2^{k+1} - 7.\end{aligned}$$
     So in this case we can simply choose $b = a + 2^{k-1}$ and $m = x + y + 1 + 2^{k-3}$.

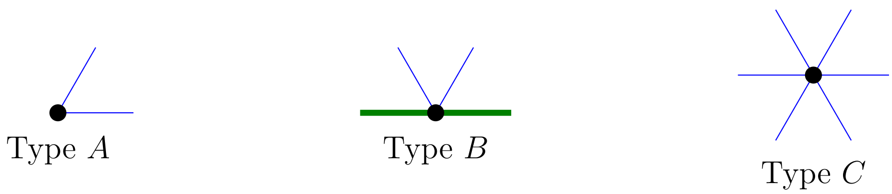
   Note here that this inductive step only works when $k \ge 3$ (otherwise $m = x + y + 1 + 2^{k-3}$ is not an integer). $\square$

5. *An equilateral triangle is partitioned into smaller equilateral triangular pieces. Prove that two of the pieces are the same size.*

   Solution: For the purpose of this proof, we will consider a *vertex* to be any point which is a corner of at least one of the triangular pieces. Define an *edge* to be any line segment between two vertices, which is part of a side of a triangular piece but does not pass through any other vertex. Note that each vertex must be one of the following types:

   - Type A: incident with only 2 edges at 60°.
   - Type B: incident with exactly 4 edges at angles $60^\circ, 60^\circ, 60^\circ, 180^\circ$.
   - Type C: incident with exactly 6 edges forming six 60° angles.

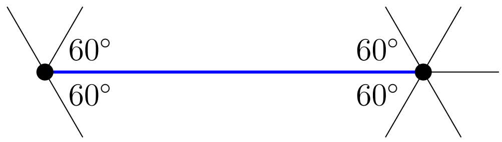
   There can be no other types of vertex, because all the angles must be 60° or 180° (or 300° in the corners of the original large triangle). We will now colour each edge-end either green or blue, as shown in the diagram (the green edge-ends are also a bit thicker).



An edge-end is coloured green if it touches a Type B vertex along the $180°$ angle, otherwise it is coloured blue. Now observe that there must be exactly 3 vertices of type A (the three corners of the original large triangle before it was partitioned). If there are x vertices of type B and y vertices of type C then this makes a total of

$$(6 + 2x + 6y) \text{ blue ends, but only } (2x) \text{ green ends.}$$

Note also that the 6 edges with an endpoint at a type A vertex must all have their other end being green. There are more blue edge-ends than green edge-ends, so it can't be the case that every blue edge-end is connected to a green edge-end. There must therefore be an edge such that both of its ends are blue. Since neither endpoint of such an edge is a type A vertex, we can conclude that all four of the angles at the endpoints of this double blue-ended edge must be $60°$.



Therefore this edge is a shared side of two triangular pieces. These two triangular pieces must therefore be the same size. □

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