New Zealand Mathematical Olympiad Committee

NZMO Round Two 2020 — Instructions

Exam date: 18^{th} September

General Instructions

1. You have 3 hours to work on the exam.
2. There are 5 problems, each worth equal marks. You should attempt all 5 problems. You may work on them in any order.
3. Geometrical instruments (ruler and compasses) may be used. Calculators, phones, computers and electronic devices of any sort are not permitted.
4. Write your solutions on the paper provided. Let the supervisor know if you need more. Write your name at the top of every page as you start using it.
5. Full written solutions — not just answers — are required, with complete proofs of any assertions you make. Your marks will depend on the clarity of your mathematical presentation. Work in rough first, and then draft a neat final version of your best attempt.
6. Make sure you fill in your details on the cover sheet. Hand in all of your rough work, in addition to your neat solutions.
7. At the end of the exam, remain seated quietly until all scripts have been collected and the supervisor indicates that you are free to move.
8. You may not take the question paper from the exam room.
9. The contest problems are to be kept confidential until they are posted on the NZMOC webpage www.mathsolympiad.org.nz. Do not disclose or discuss the problems online until this has occurred.
10. **Do not turn over until told to do so**.

NZMO Round Two 2020 — Problems

Exam date: 18^{th} September

- There are 5 problems. You should attempt to find solutions for as many as you can.
- Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- Read and follow the "General instructions" accompanying these problems.

Problems

1. Let $P(x) = x^3 - 2x + 1$ and let $Q(x) = x^3 - 4x^2 + 4x - 1$. Show that

$$\text{if } P(r) = 0 \text{ then } Q(r^2) = 0.$$

2. Find the smallest positive integer N satisfying the following three properties.
   - N leaves a remainder of 5 when divided by 7.
   - N leaves a remainder of 6 when divided by 8.
   - N leaves a remainder of 7 when divided by 9.
3. There are 13 marked points on the circumference of a circle with radius 13. Prove that we can choose three of the marked points which form a triangle with area less than 13.
4. Let Γ_1 and Γ_2 be circles internally tangent at point A, with Γ_1 inside Γ_2. Let BC be a chord of Γ_2 which is tangent to Γ_1 at point D. Prove that line AD is the angle bisector of $\angle BAC$.
5. A sequence of As and Bs is called *antipalindromic* if writing it backwards, then turning all the As into Bs and vice versa, produces the original sequence. For example $ABBAAB$ is antipalindromic. For any sequence of As and Bs we define the *cost* of the sequence to be the product of the positions of the As. For example, the string $ABBAAB$ has cost $1 \cdot 4 \cdot 5 = 20$. Find the sum of the costs of all antipalindromic sequences of length 2020.