New Zealand Mathematical Olympiad Committee

NZMO Round Two 2021 — Instructions

Exam date: 4^{th} October

General Instructions

1. You have 3 hours to work on the exam.
2. There are 5 problems, each worth equal marks. You should attempt all 5 problems. You may work on them in any order.
3. A ruler and compass may be used, but other geometrical instruments (e.g. protractors) may not be used. Calculators, phones, computers and electronic devices of any sort are not permitted.
4. Write your solutions on blank paper. There is no limit to how many answer pages you may use. Write your name at the top of every page as you start using it.
5. Full written solutions — not just answers — are required, with complete proofs of any assertions you make. Your marks will depend on the clarity of your mathematical presentation. Work in rough first, and then draft a neat final version of your best attempt.
6. **All** pages, including your rough work, are to be submitted.
7. At the end of the exam, scan your solution pages and submit them as quickly as possible. If submissions are delayed for any reason, email info.nzmoc@gmail.com immediately.
8. The contest problems are to be kept confidential until they are posted on the NZMOC webpage www.mathsolympiad.org.nz. Any discussion of the problems before this has occured is strictly forbidden.
9. **Do not turn over until the start time of your exam**.

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NZMO Round Two 2021 — Problems

Exam date: 4^{th} October

- There are 5 problems. You should attempt to find solutions for as many as you can.
- Solutions (that is, answers with justifications) and not just answers are required for all problems, even if they are phrased in a way that only asks for an answer.
- Read and follow the "General instructions" accompanying these problems.

Problems

1. Let $ABCD$ be a convex quadrilateral such that $AB + BC = 2021$ and $AD = CD$. We are also given that
$$\angle ABC = \angle CDA = 90^\circ$$
Determine the length of the diagonal BD.

2. Prove that
$$x^2 + \frac{8}{xy} + y^2 \geq 8.$$
for all positive real numbers x and y.

3. Let $\{x_1, x_2, x_3, \ldots, x_n\}$ be a set of n distinct positive integers, such that the sum of any 3 of them is a prime number. What is the maximum value of n?

4. Let AB be a chord of circle Γ. Let O be the centre of a circle which is tangent to AB at C and internally tangent to Γ at P. Point C lies between A and B. Let the circumcircle of triangle POC intersect Γ at distinct points P and Q. Prove that $\angle AQP = \angle CQB$.

5. Find all pairs of integers x, y such that
$$y^5 + 2xy = x^2 + 2y^4.$$