New Zealand Mathematical Olympiad Committee

Maths Workshop

August 2021

Problems

1. For which real numbers k does the following equation have at least one real solution?

$$\frac{x^2 + 3x + 2}{2x + 5} = k$$

2. In quadrilateral $ABCD$ we have $\angle DAB = \angle CDA = 60^\circ$ and sidelength: $BC = 5$, $CD = 6$ and $AD = 9$. Determine length AB.

3. Find all positive integers n and m such that

$$m! + 5 = n^3.$$

4. Sixteen ($16 = 2^4$) teams participate in a single-elimination tournament of four rounds. That is, each team plays some other team in the first round, and the winners advance to the second round, and so forth. Suppose the teams are currently ranked best to worst from #1 to #16 and the higher-ranked team always wins in every game. The tournament designers, instead of using the rankings, chose the team match ups completely randomly. What is the probability that teams $\#1 - \#8$ all advance to the quarterfinals (second round), teams $\#1 - \#4$ all advance to the semifinals (third round), and teams #1 and #2 meet in the final (fourth round)?

5. Let ABC be any acute-angled triangle. Suppose that points D, E and F lie on sides BC, CA and AB respectively such that $BD = DF$ and $CD = DE$. Prove that the circumcentre of triangle AEF lies on the angle bisector of $\angle EDF$.

6. Let $a_1, a_2, a_3, \ldots, a_n$ be any sequence of distinct positive integers. Prove that there exists an integer n such that

$$\frac{a_1}{1} + \frac{a_2}{4} + \frac{a_3}{9} + \frac{a_4}{16} + \cdots + \frac{a_n}{n^2} > 2021.$$